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ON THE SECOND HANKEL DETERMINANT OF AREALLY MEAN p-VALENT FUNCTIONS

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ABSTRACT. In this paper we determine the growth rate of the second Hankel determinant of an areally mean p-valent function. This result both extends and unifies previously known results concerning this problem.

I. Introduction and statement of results. Let f be regular in $\gamma = \{z: |z| < 1\}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The qth Hankel determinant of f is defined for $q \geqslant 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & \cdots\cdots & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n+q-1} & \cdots\cdots & a_{n+2q-2} \end{vmatrix}.$$

If $n(\omega)$ is the number of roots in γ of the equation $f(z) = \omega$, f is said to be areally mean p-valent in γ [1] if for all $R > 0$,

$$W(R, f) = \frac{1}{\pi} \int_0^{2\pi} \int_0^R n(\rho e^{i\theta}) \rho \, d\rho \, d\theta \leqslant pR^2.$$

As usual, f is normalized so that $\max\{|a_k|: 0 \leqslant k \leqslant [p]\} = 1$, and the class of normalized areally mean p-valent functions is denoted by S_p.

The problem of determining the rate of growth of $H_q(n)$ as $n \longrightarrow \infty$ when $f \in S_p$ is well known. Ch. Pommerenke [9] has shown that for $p \geqslant 1$, $H_q(n) = O(1)\, n^{k\sqrt{q} - q/2}$ where $k = 16p\sqrt{p}$. The present authors have shown [8] that if $q \geqslant 2$ and $p \geqslant 2(q-1)$, then $H_q(n) = O(1) n^{2pq - q^2}$, where the exponent is best possible. For strictly univalent functions, Pommerenke [10] has shown that for $q \geqslant 2$, $H_q(n) = O(1) n^{-(1/2+\beta)q + 3/2}$, where $\beta > 1/4000$. In particular, $H_2(n) = O(1) n^{1/2 - 2\beta}$. On the other hand, W. K. Hayman [4] has shown that $H_2(n) = o(1) n^{1/2}$ when $f \in S_1$, and that this is best possible.

It is clear that the known results concerning this problem are incomplete,

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in the sense that given q, best possible growth rates for $H_q(n)$ are known only for certain values of p. In this paper we shall examine the behavior of $H_2(n)$ when $f \in S_p$. We prove

THEOREM 1. *Let $f \in S_p$. Then, as $n \to \infty$,*

$$H_2(n) = a_n a_{n+2} - a_{n+1}^2 = \begin{cases} o(1) n^{-1}, & 0 < p < 1/4, \\ o(1) n^{2p-3/2}, & 1/4 < p < 5/4, \\ O(1) n^{4p-4}, & p > 5/4. \end{cases}$$

If $p > 5/4$ and $\lim_{r\to 1}(1-r)^{2p}M(r, f) = 0$, then $H_2(n) = o(1)n^{4p-4}$.

In the opposite direction we have

THEOREM 2. *Given any positive sequence $\{\epsilon_n\}$ with $\lim_{n\to\infty}\epsilon_n = 0$, there exists $f \in S_p$ such that for infinitely many n,*

$$|H_2(n)| > \begin{cases} \epsilon_n n^{-1}, & 0 < p < 1/4, \\ \epsilon_n n^{2p-3/2}, & 1/4 < p < 5/4. \end{cases}$$

In addition, if $p > 5/4$ and $f \in S_p$ satisfies $\alpha = \lim_{r\to 1}(1-r)^{2p}M(r, f) > 0$, then $\lim_{n\to\infty}|H_2(n)|/n^{4p-4} = \alpha^2(2p-1)/\Gamma(2p)^2$.

Theorem 2 shows that the results of Theorem 1 are best possible, and also that when $p > 5/4$, $O(1)$ cannot in general be replaced by $o(1)$. These results essentially solve Problem 6.14′ of [2] when $q = 2$.

II. **Preliminary results.** With $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ and y any complex number, we set $\Delta_0(n+2, y, f) = a_{n+2}$, $\Delta_1(n+1, y, f) = a_{n+1} - y a_{n+2}$, and $\Delta_2(n, y, f) = a_n - 2y a_{n+1} + y^2 a_{n+2}$, so that

$$H_2(n) = \Delta_2(n, y, f)\Delta_0(n+2, y, f) - \Delta_1(n+1, y, f)^2. \tag{2.1}$$

We shall estimate the various terms in (2.1) by combining two methods due originally to W. K. Hayman [3], [4]. For the sake of brevity, we shall refer to the existing literature whenever possible.

If $z_1 \in \gamma$ and $z = \rho e^{i\theta}$, Cauchy's theorem gives that

$$|\Delta_2(n, z_1, zf')| \leq \frac{1}{2\pi\rho^{n+1}} \int_0^{2\pi} |z - z_1|^2 |f'(z)| \, d\theta,$$

and upon integrating from $\rho = 1 - 3/n$ to $\rho = 1 - 2/n$, we find that

$$\frac{|\Delta_2(n, z_1, zf')|}{n} = O(1) \int_0^{2\pi} \int_{1-3/n}^{1-2/n} |z - z_1|^2 |f'(z)| \rho \, d\rho \, d\theta. \tag{2.2}$$

Henceforth we assume that n is fixed and that z_1 has been chosen so that

$|z_1| = n/(n+1)$, $|f(z_1)| = M(n/(n+1), f)$. We also set $M_1 = |f(z_1)|$, $M_k = e^{1-k}M_1$, $k \geqslant 1$.

Technical considerations dictate that we now proceed in two similar yet different ways. We first divide $E = \{z\colon\ 1 - 3/n \leqslant \rho \leqslant 1 - 2/n\}$ into disjoint subsets $E_k = \{z \in E\colon M_{k+1} \leqslant |f(z)| < M_k\}$. Upon using the techniques of [3] (see also [8, p. 508]) we conclude that

$$\frac{|\Delta_2(n, z_1, zf')|}{n} = O(1) \sum_{k=1}^{\infty} \left\{ \int_0^{2\pi} \int_{1-3/n}^{1-2/n} |z - z_1|^4 G_k(|f(z)|) \rho \, d\rho \, d\theta \right\}^{1/2}, \tag{2.3}$$

where $G_k(R) = M_k^2 R^2/(M_k^2 + R^2)$.

Following Hayman [4], we now introduce a slightly different method. Choosing $\lambda > 2$, applying the Schwarz inequality to (2.2), and noting that

$$\int_0^{2\pi} \int_{1-3/n}^{1-2/n} |f'(z)|^2 (1 + |f(z)|^\lambda)^{-1} \rho \, d\rho \, d\theta \leqslant A(\lambda)$$

(see [4, p. 81]), we deduce that

$$\begin{aligned} &\frac{|\Delta_2(n, z_1, zf')|}{n} \\ &\qquad \leqslant A(\lambda) \left\{ n^{-1/2} + \left(\int_0^{2\pi} \int_{1-3/n}^{1-2/n} |z - z_1|^4 |f(z)|^\lambda \rho \, d\rho \, d\theta \right)^{1/2} \right\}. \end{aligned} \tag{2.4}$$

(As usual, $A_1, A_2, \ldots$ denote absolute constants, while $A(x, y, \ldots)$ denotes a constant depending only on $x, y, \ldots$.)

The estimate (2.3) will be used when $p > 5/4$, and (2.4) will be used when $1/4 < p < 5/4$.

III. Estimate for (2.4). Applying [6, Lemma 2] and [4, Lemma 3] to $\int_0^{2\pi} |\rho e^{i\theta} - z_1|^4 |f(\rho e^{i\theta})|^\lambda \, d\theta$, we find that

$$\int_0^{2\pi} |\rho e^{i\theta} - z_1|^4 |f(\rho e^{i\theta})|^\lambda d\theta \leqslant A(p, \lambda) + A_1(J_1(\rho) + J_2(\rho)), \tag{3.1}$$

where

$$J_1(\rho) = \int_{1/2}^{\rho} \int_0^{2\pi} |re^{i\theta} - z_1|^2 |f(re^{i\theta})|^\lambda r \log \rho/r \, d\theta \, dr,$$

$$J_2(\rho) = \int_{1/2}^{\rho} \int_0^{2\pi} |re^{i\theta} - z_1|^4 |f'(re^{i\theta})|^2 |f(re^{i\theta})|^{\lambda-2} r \log \rho/r \, d\theta \, dr.$$

The essential part of our proof consists of deriving appropriate estimates for $J_1(\rho)$ and $J_2(\rho)$. We begin with $J_2(\rho)$.

LEMMA 1. *Let $f \in S_p$, $\lambda > 2$, k a positive integer. Then for any a satisfying $0 < a \leq k$, we have*

$$\int_{1/2}^{\rho}\int_0^{2\pi} |re^{i\theta} - z_1|^{2k} |f'(re^{i\theta})|^2 |f(re^{i\theta})|^{\lambda-2} r \log \rho/r \, d\theta \, dr$$

$$\leq A(p, \lambda, a) \begin{cases} \left(\dfrac{n^{2p}}{M_1}\right)^{a^2/2p} & \text{if } 1 < 2p\lambda < 2a + 1, \\ \left(\dfrac{n^{2p}}{M_1}\right)^{a^2/2p} \min \{M_1, (1-\rho)^{-2p}\}^{(2p\lambda-2a-1)/2p} & \\ & \text{if } 2p\lambda > 2a + 1. \end{cases}$$

PROOF. Divide the range of integration into subsets $F_j = \{re^{i\theta}: 1/2 \leq r < \rho, M_{j+1} \leq |f(re^{i\theta})| < M_j\}$; also note that $\log \rho/r < 2(1 - r)$. Following Hayman [4], we suppose that $M_{j+1} \geq M(1/2, f)$ for at least one value of j. (The opposite case is trivial.) Since $0 < a \leq k$, $|re^{i\theta} - z_1|^{2k} \leq A|re^{i\theta} - z_1|^{2a}$, and so [5, Theorem 1]

$$|re^{i\theta} - z_1|^{2k} \leq A(p, a)(n^{2p}/M_1)^{a^2/2p}(1 - r)^{-1} |f(re^{i\theta})|^{(-2a-1)/2p}.$$

Therefore

$$\iint_{F_j} |re^{i\theta} - z_1|^{2k} |f'(re^{i\theta})|^2 |f(re^{i\theta})|^{\lambda-2} \log \rho/r \, d\theta \, dr$$

$$\leq A(p, a)\left(\frac{n^{2p}}{M_1}\right)^{a^2/2p} M_j^{-\epsilon}$$

where $\epsilon = -\lambda + (2a + 1)/2p$.

If $\epsilon > 0$ (i.e. $2p\lambda < 2a + 1$), we choose constants b and $A(p)$ with $0 < b \leq M(1/2, f) \leq A(p)$, we define $j_0 = j_0(n) = \max\{j: M_{j+1} \geq M(1/2, f)\}$, and we conclude that $b \leq M_{j_0+1} \leq A(p)$. With $F'' = \{re^{i\theta}: |f(re^{i\theta})| \leq M_{j_0+1}\}$, it follows easily from the definition of the class S_p that

$$\iint_{F''} |re^{i\theta} - z_1|^{2k} |f'(re^{i\theta})|^2 |f(re^{i\theta})|^{\lambda-2} \log \rho/r \, dr \, d\theta \leq A(p, \lambda, k).$$

The above remarks therefore imply

$$\int_{1/2}^{\rho}\int_0^{2\pi} |re^{i\theta} - z_1|^{2k} |f'(re^{i\theta})|^2 |f(re^{i\theta})|^{\lambda-2} r \log \rho/r \, d\theta \, dr$$

$$= \iint_{F''} + \sum_{j=1}^{j_0} \iint_{F_j}$$

$$\leq A(p, \lambda, k) + A(p, a)\left(\frac{n^{2p}}{M_1}\right)^{a^2/2p} \sum_{j=1}^{j_0} M_j^{-\epsilon}.$$

Since $\epsilon > 0$, $\Sigma_{j=1} M_j^{-\epsilon} \leqslant b^{-\epsilon}\Sigma_{j=1}^{\infty} e^{-\epsilon j} < \infty$, and the lemma follows.

If $\epsilon < 0$ (i.e. $2p\lambda > 2a + 1$), we divide the range of integration into subsets $E_j = \{re^{i\theta}: 1/2 \leqslant r < \rho, N_{j+1} \leqslant |f(re^{i\theta})| < N_j\}$, where $N_1 = \min\{M_1, A(p)(1-\rho)^{-2p}\}$, $N_j = e^{1-j}N_1$. As above, we conclude that

$$\iint_{E_j} |re^{i\theta} - z_1|^{2k} |f'(re^{i\theta})|^2 |f(re^{i\theta})|^{\lambda-2} \log \rho/r \, dr d\theta$$

$$\leqslant A(p, a, \lambda)(n^{2p}/M_1)^{a^2/2p} N_j^{-\epsilon}.$$

Upon summing from $j = 1$ to ∞ and using the fact that

$$\sum_{j=1}^{\infty} N_j^{-\epsilon} \leqslant A(p, \lambda, a) N_1^{-\epsilon},$$

we arrive at the conclusion of the lemma.

We now estimate $J_1(\rho)$.

LEMMA 2. *Let $f \in S_p$, $\lambda > 2$, and $0 < \alpha \leqslant 1$. Suppose that $1 < 2p\lambda < 2\alpha + 3$ and $2p\lambda \neq 2\alpha + 1$, $2p\lambda \neq 3$. Then $J_1(\rho) \leqslant A(p, \alpha, \lambda)\,(n^{2p}/M_1)^{\alpha^2/2p}$.*

PROOF. Using [6, Lemma 2] and [4, Lemma 3], we see that

$$K(r) \leqslant A_1 + 16K_1(r) + 4\lambda^2 K_2(r),$$

where

$$K(r) = \int_0^{2\pi} |re^{i\theta} - z_1|^2 |f(re^{i\theta})|^{\lambda} d\theta,$$

$$K_1(r) = \int_{1/2}^{r}\int_0^{2\pi} |f(te^{i\theta})|^{\lambda}(1-t)\,d\theta\, dt,$$

$$K_2(r) = \int_{1/2}^{r}\int_0^{2\pi} |te^{i\theta} - z_1|^2 |f(te^{i\theta})|^{\lambda-2} |f'(te^{i\theta})|^2 (1-t)\,d\theta\, dt.$$

From Lemma 1 (with $k = 1$), we find that

$$K_2(r) \leqslant A(p, \lambda, \alpha) \begin{cases} (n^{2p}/M_1)^{\alpha^2/2p}, & \text{if } 1 < 2p\lambda < 2\alpha + 1, \\ (n^{2p}/M_1)^{\alpha^2/2p} \min\{M_1, (1-r)^{-2p}\}^{(2p\lambda - 2\alpha - 1)/2p}, & \\ & \text{if } 2p\lambda > 2\alpha + 1. \end{cases}$$

Also, from [1, Theorem 3.2], we have $\int_0^{2\pi} |f(te^{i\theta})|^{\lambda} d\theta \leqslant A(p, \lambda)(1-t)^{1-2p\lambda}$, provided $2p\lambda > 1$. Thus

$$K_1(r) \leqslant A(p, \lambda) \begin{cases} 1, & \text{if } 1 < 2p\lambda < 3, \\ (1-r)^{3-2p}, & \text{if } 2p\lambda > 3. \end{cases}$$

Hence

$$K(r) \leqslant A(p, \lambda, \alpha)\left(\frac{n^{2p}}{M_1}\right)^{\alpha^2/2p} \begin{cases} 1, & \text{if } 1 < 2p\lambda < 2\alpha + 1, \\ (1-r)^{1+2\alpha-2p\lambda}, & \text{if } 2\alpha + 1 < 2p\lambda < 3 \\ & \text{or } 2p\lambda > 3. \end{cases}$$

Since $J_1(\rho) = \int_{1/2}^{\rho} K(r) \log \rho/r \, dr$, the lemma now follows immediately.

We can now estimate $\Delta_2(n, z_1, zf')$. Choose a such that $0 < a \leqslant 2$, and put $\alpha = a/2$. Then Lemmas 1 and 2 imply that

$$J_1(\rho) + J_2(\rho) \leqslant A(p, \lambda, a)(n^{2p}/M_1)^{a^2/2p} \tag{3.2}$$

provided $1 < 2p\lambda < 2a + 1$, $2p\lambda \neq a + 1$, $2p\lambda \neq 3$. Upon combining (2.4), (3.1), and (3.2), we find that

$$\begin{aligned} \frac{|\Delta_2(n, z_1, zf')|}{n} &\leqslant A(\lambda)\left\{n^{-1/2} + A(p, a, \lambda)\left(\frac{n^{2p}}{M_1}\right)^{a^2/4p}\left\{\int_{1-3/n}^{1-2/n} d\rho\right\}^{1/2}\right\} \\ &\leqslant A(p, a, \lambda) n^{-1/2}\left(\frac{n^{2p}}{M_1}\right)^{a^2/4p} \end{aligned} \tag{3.3}$$

for any a such that $0 < a \leqslant 2$, $1 < 2p\lambda < 2a + 1$, $2p\lambda \neq a + 1$, $2p\lambda \neq 3$.

IV. **Proof of Theorem 1 when $0 < p < 1/4$ or $1/4 < p < 5/4$.** If $0 < p < 1/4$, then [11] $a_n = o(1)n^{-1/2}$, and so trivially $H_2(n) = o(1)n^{-1}$. Now suppose $1/4 < p < 5/4$. We first note that for $p > 1/4$,

$$|a_n| \leqslant A(p) n^{-1/2} M_1^{1-1/4p}, \tag{4.1}$$

a result proved exactly as in [4] in the case $p = 1$. Also, with $z_1 = e^{i\theta_n} n/(n + 1)$, we have

$$\Delta_2(n, e^{i\theta_n}, f) = n^{-1}\Delta_2(n, z_1, zf') + (n + 1)^{-2} e^{2i\theta_n} a_{n+2}.$$

Combining this with (3.3), (4.1), and the fact that $1/4 < p < 5/4$, we see that

$$\Delta_2(n, e^{i\theta_n}, f) \leqslant A(p, a, \lambda) n^{-1/2}(n^{2p}/M_1)^{a^2/4p} \tag{4.2}$$

with a as before.

We now prove that $H_2(n) = o(1)n^{2p-3/2}$ when $1/4 < p < 5/4$. It follows from (4.1) and (4.2) that

$$\Delta_2(n, e^{i\theta_n}, f)\Delta_0(n + 2, e^{i\theta_n}, f) \leqslant A(p, a, \lambda) n^{2p-3/2}(n^{2p}/M_1)^{\delta},$$

where $\delta = (a^2 + 1 - 4p)/4p$. Choose $a = (2p\lambda - 1)/2 + \epsilon$, where $\lambda > 2$ and $\epsilon > 0$ are chosen such that all previous restrictions involving a are satisfied, and also such that $\delta < 0$. (Elementary computations verify that such a choice is possible; the fact that $1/4 < p < 5/4$ is essential here.)

We next note [6] that

$$(4.3)\qquad |\Delta_1(n+1, e^{i\theta_n}, f)| \leqslant A(p,\lambda)\begin{cases} n^{2p-2\sqrt{p}}, & 1/4 < p < 1,\\ n^{2p-2}, & p \geqslant 1,\end{cases}$$

and so

$$|\Delta_1(n+1, e^{i\theta_n}, f)|^2 = o(1)n^{2p-3/2},$$

where again we have used $1/4 < p < 5/4$. We thus conclude from (2.1) and the above remarks that

$$|H_2(n)|/n^{2p-3/2} \leqslant A(p)(M_1/n^{2p})^{-\delta} + o(1).$$

If $M_1 = M(n/(n+1), f) = o(1)n^{2p}$, then (since $-\delta > 0$) $H_2(n) = o(1)n^{2p-3/2}$. If $M_1 \neq o(1)n^{2p}$, it is well known [1] that $\lim_{r\to 1}(1-r)^{2p}M(r,f) > 0$. From (3.1) and Lemmas 1 and 2, it follows that with $\lambda > 2$ fixed, $\int_0^{2\pi}|\rho e^{i\theta} - z_1|^4|f(\rho e^{i\theta})|^\lambda d\theta$ is uniformly bounded for $0 < \rho < 1$. We now use exactly the same technique as does Hayman [4, p. 90] to conclude that $\Delta_2(n, z_1, zf') = o(1)n^{1/2}$, and then as above we deduce that

$$H_2(n) = o(1)n^{2p-3/2}.$$

This completes the proof of Theorem 1 in the case $1/4 < p < 5/4$.

V. Estimate for (2.3). We now assume $p > 5/4$. Our method is essentially that of [8], the major difference being that since we are dealing with the specific case $q = 2, p > 5/4$, we can make more efficient use of the two-point modulus bound than was possible in [8]. In view of the technical nature of this modification, we shall merely indicate the sort of changes to be made in [8]. Verification of the complete details will be left to the interested reader.

Recalling that $G_k(R) = M_k^2R^2/(M_k^2 + R^2)$, we see upon using [6, Lemma 2] and [4, Lemma 3] that

$$\int_0^{2\pi} |\rho e^{i\theta} - z_1|^4 G_k(|f(\rho e^{i\theta})|)d\theta$$

can be estimated in terms of seven integrals (see [8, p. 511]), of which the most troublesome is

$$(5.1)\qquad \int_{1/2}^{\rho}(1-t)\int_0^{2\pi}|te^{i\theta} - z_1|^2 G_k(|f(te^{i\theta})|)d\theta\, t\,dt.$$

Reapplying [6, Lemma 2] and [4, Lemma 3], we can estimate the inner integral of (5.1) in terms of seven more integrals, the most troublesome being

$$(5.2)\quad \int_{1/2}^{t}(1-t_1)\int_0^{2\pi}|t_1e^{i\theta} - z_1|^2|f'(t_1e^{i\theta})|^2 \frac{M_k^4(M_k^2 - R^2)}{(M_k^2 + R^2)^3}d\theta\, t_1dt_1.$$

In order to estimate this integral we first replace the range of integration by $\Omega = \{t_1e^{i\theta}\colon\ 1/2 \leq t_1 < t,\ |f(t_1e^{i\theta})| \leq B_1\}$, where $B_1 = \min\{M_k, A(p)(1-t)^{-2p}\}$. We now divide Ω into subsets $\Omega_m = \{t_1e^{i\theta}\in\Omega\colon B_{m+1} \leq |f(t_1e^{i\theta})| < B_m\}$, where $B_m = e^{1-m}B_1$. An application of the two-point modulus bound (put $a = b = 1$ in [5, Theorem 1]) allows us to conclude that integration over Ω_m contributes at most $A(p)(n^{2p}/M_1)^{1/2p}B_m^{(4p-3)/2p}$, and upon summing from $m = 1$ to ∞, we conclude that (5.2) is bounded above by $A(p)(n^{2p}/M_1)^{1/2p}\min\{M_k^{2-3/2p}, (1-t)^{3-4p}\}$. Combining this estimate with the technique of [8, p. 517], we see that integration of (5.2) contributes at most $A(p)e^{-(4p-5)k/2p}(n^{2p}/M_1)^{1/2p}M_1^{(4p-5)/2p}$ to (5.1).

After re-examining the arguments of [8] in light of the changes suggested above, we find that

$$\int_0^{2\pi}|\rho e^{i\theta} - z_1|^4G_k(|f(\rho e^{i\theta})|)\,d\theta \leq A(p)e^{-(4p-5)k/4p}(n^{2p}/M_1)^{4/2p}n^{4p-5}.$$

Summing these estimates (as required by (2.3)), we find that

$$\frac{|\Delta_2(n, z_1, zf')|}{n} \leq A(p)\left(\frac{n^{2p}}{M_1}\right)^{4/4p} n^{2p-3}. \tag{5.3}$$

The important point concerning this method is that if $p > 5/4$, the presence of the convergence factor $e^{-(4p-5)k/2p}$ allows us to sum from $k = 1$ to ∞ and obtain (5.3).

VI. **Proof of Theorem 1 when** $p > 5/4$. The estimate (5.3) leads, in the same manner as in the case $1/4 < p < 5/4$, to the estimate

$$|\Delta_2(n, e^{i\theta}n, f)| \leq A(p)\,(n^{2p}/M_1)^{4/4p}n^{2p-3}.$$

Upon combining this with (4.1), we find that

$$|\Delta_2(n, e^{i\theta}n, f)\Delta_0(n+2, e^{i\theta}n, f)| \leq A(p)n^{4p-4}(M_1/n^{2p})^{4p-5/4p}. \tag{6.1}$$

From (4.3) we have $|\Delta_1(n+1, e^{i\theta}n, f)|^2 \leq A(p)n^{4p-4}$, and upon combining this with (6.1) and (2.1), we conclude that $H_2(n) = O(1)n^{4p-4}$, as required. (Here we have used the facts $p > 5/4$ and $M_1 \leq A(p)n^{2p}$.)

If $M(r, f) = o(1)(1-r)^{-2p}$, it is clear that $o(1)$ replaces $O(1)$ in (6.1), and from [7, Theorem 1] we have

$$\Delta_1(n+1, e^{i\theta}n, f) = o(1)n^{2p-2} \quad \text{for } p > 1.$$

Thus $H_2(n) = o(1)n^{4p-4}$.

If $\alpha = \lim_{r\to 1}(1-r)^{2p}M(r, f) > 0$, it follows from [7, Theorem 4] that for $p > 5/4$,

$$\lim_{n\to\infty} \frac{|H_2(n)|}{n^{4p-4}} = \frac{\alpha^2(2p-1)}{\Gamma(2p)^2}. \tag{6.2}$$

Thus $O(1)$ cannot in general be replaced by $o(1)$ when $p > 5/4$.

VII. **Examples when $p < 5/4$.** Since (6.2) applies when $p > 5/4$, we see that it remains only to prove the first statement of Theorem 2. Put $f_0(z) = 2^{2p}\pi(1-z)^{-2p} = \Sigma_{n=0}^{\infty} A_n z^n$ and $\varphi(z) = \Sigma_{n=2}^{\infty} b_n z^n$, where $\{b_n\}_2^{\infty}$ is any sequence of nonnegative numbers with $\Sigma_{n=2}^{\infty} b_n \leqslant 1$, $\Sigma_{n=2}^{\infty} nb_n^2 \leqslant p$. In [7] it is shown that if $p \geqslant 1/2$, the function f given by

$$f(z) = f_0(z) + \varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

is areally mean p-valent.

Suppose now that $\{\epsilon_n\}$ is as in Theorem 2, and choose $\{b_n\}$ such that for some subsequence $\{n_k\}$, $b_{n_k} = b_{n_k+2} = 0$, $b_{n_k+1} = \epsilon_{n_k} n_k^{-1/2}$. Direct calculation, in which we use the fact that $a_n = A_n + b_n$, shows that

$$H_2(n_k, f) = H_2(n_k, f_0) - 2A_{n_k+1}b_{n_k+1} - b_{n_k+1}^2. \tag{7.1}$$

Since $\lim_{r\to 1}(1-r)^{2p}M(r, f_0) > 0$, it follows from (6.2) that

$$\lim_{n_k\to\infty} \frac{H_2(n_k, f_0)}{n_k^{2p-3/2}} = 0,$$

where we have used strongly the fact that $p < 5/4$. Also, $b_{n_k+1}^2/n_k^{2p-3/2} = \epsilon_{n_k}^2/n_k^{(4p-1)/2}$, and

$$\frac{A_{n_k} b_{n_k+1}}{n_k^{2p-3/2}} = \frac{2^{2p}\pi\epsilon_{n_k}}{\Gamma(2p)}(1 + o(1)),$$

since $A_n \sim 2^{2p}\pi n^{2p-1}/\Gamma(2p)$. Theorem 2 (for $p \geqslant 1/2$) now follows immediately from these estimates and (7.1).

In order to prove Theorem 2 for $1/4 < p < 1/2$, we use the same technique as above, except that we are forced to alter the function f slightly. Given p with $1/4 < p < 1/2$, construct $F \in S_p$ as follows (see [3], [4]). Put $g(z) = (1-z)^{-1}$, $\chi = 2/\cos p\pi$, $G(z) = g(z)^{2p} + \chi$, and $F(z) = G(z) + \varphi(z)$, where φ is as before. Clearly all we need do to prove Theorem 2 is to show that $F \in S_p$.

Note first that G maps γ conformally into the sector

$$E = \{\omega\colon |\arg(\omega - \chi)| < p\pi\}.$$

Put $\omega = \chi + te^{i\theta}$, so that $|\omega|^2 = t^2 + x^2 + 2xt\cos\theta$. If $\omega \in E$, it follows from the definition of χ that $|\omega|^2 \geqslant (t+2)^2$, and so $t \leqslant |\omega| - 2$.

Set $E_R = E \cap \{\omega\colon |\omega| \leqslant R\}$, and let $A(R)$ be the area of E_R. If $R < \chi$, then $A(R) = 0$, while if $R > \chi$, $A(R) < p\pi(R-2)^2$. The argument employed by Hayman [3] now shows that $F \in S_p$. As noted above, this proves Theorem 2 for $1/4 < p < 1/2$. In conclusion, we note that the example given in [1, p. 49] shows that Theorem 2 also holds for $0 < p < 1/4$.

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